Indian Statistical Institute, Bangalore
M. Math First Year
Second Semester - Complex Analysis

Back Paper Exam | Date: June 13, 2018
Maximum marks: 100 | Duration: 3 hours

1. Let $\mathbb{D}$ be the open unit disc, S^1 its boundary.
   (a) Show that, for $\alpha \in S^1, \beta \in \mathbb{D}$, the rational function $z \to \alpha \frac{z-\beta}{1-\bar{\beta}z}$ maps $\mathbb{D}$ biholomorphically onto $\mathbb{D}$ and S^1 bijectively onto S^1.
   (b) Show that the functions defined in (a) form a group under composition. [10 + 10 = 20]

2. Let Ω be a simply connected domain such that $0 \in \Omega$. Let $f : \Omega \to \mathbb{D}$ be a holomorphic function such that $f(0) = 0$. If f is not onto $\mathbb{D}$, then show that there is a holomorphic function $g : \Omega \to \mathbb{D}$ such that $g(0) = 0$ and $|g'(0)| > |f'(0)|$. [20]

3. Let $0 < r < R$ and let $\Omega = \{z \in \mathbb{C} : r < |z| < R\}$. If $f : \Omega \to \mathbb{C}$ is holomorphic then show that there is a series representation $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ (which converges for all $z \in \Omega$). [20]

4. (a) If Ω is a simply connected planar domain then show that every non-vanishing holomorphic function on Ω has a holomorphic logarithm.
   (b) Give an example (with proof) to show that the assumption of simply connectedness cannot be dropped. [10 + 10 = 20]

5. (a) State and prove Rouche's Theorem.
   (b) Use Rouche's Theorem to prove that every non-zero complex polynomial of degree n has exactly n roots. [10 + 10 = 20]